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A Bivariate Local Limit Theorem

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Multivariate local limit theorems are established for sums of random variables which are in the domain of attraction of a bivariate stable law, when the concept of stability allows different scaling constants in the different components. The results cover the lattice, non-lattice, and jointly lattice and non-lattice cases.

1. Introduction

Multivariate local limit theorems for sums of random variables in the domain of attraction of a stable law have been known for a long time, and are due to Rvaceva [7], in the lattice case, and Stone [8], in the non-lattice case, respectively. However, the concept of stability in two or more dimensions has more than one interpretation, and in these references, and most of the other work in this area, the concept of stability used is the narrow one which requires the same scaling constants in each component. In [7], Rvaceva mentioned the problem of extending her results to the broader situation where the definition of stability allows different scaling constants in the different components, and it is to this question that the present paper is addressed. An examination of [7] shows that the main obstacle to such an extension is that some specific information is required about the asymptotic behaviour of $\varphi(\mathbf{t})$, the characteristic function of the limiting stable law. The general form of $\varphi(\mathbf{t})$ has been found by Resnick and Greenwood [6], and is considerably more complicated than in the narrow sense case. The crux of the present paper is the demonstration, in two dimensions, that for all $\mathbf{t}$

$$|\varphi(\mathbf{t})| \leqslant e^{-c|\mathbf{t}|^{\beta}}, \tag{1.1}$$

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where c and β are positive constants. Using this, we establish the appropriate local limit theorems in the lattice case, the non-lattice case, and a jointly lattice and non-lattice case. Our motivation in this latter case was an application of our results to the joint distribution of the first ladder epoch and ladder height in random walk, which will be given in [1]; here the joint distribution is clearly jointly lattice and non-lattice whenever the step distribution of the random walk is non-lattice.

Since our proof of (1.1) extends easily to higher dimensions, it is easy to formulate higher dimensional versions of our results, at least in the lattice and non-lattice cases.

It should be remarked that the situation we study is the special case of "matrix normalisation" in which the normalising matrix is diagonal. The more general case has been studied intensively, relevant references being Hahn and Klass [3, 4] and in particular, Griffin [2], which contains results similar to ours.

2. Results

If $\mathbf{Y}$ is a random vector in $\mathbb{R}^{(2)}$ whose distribution is not concentrated on any straight line, we say it is non-degenerate. If this is the case, it is possible to choose the coordinate system in such a way that the distribution of $\mathbf{Y}$ is either non-lattice, 1-lattice, or jointly 1-lattice and non-lattice. Here the first two terms have their usual meaning; viz. non-lattice means that $|f(\mathbf{t})| < 1$ for all $\mathbf{t} \neq \mathbf{0}$, where $f(\mathbf{t}) = E(e^{i\mathbf{t}\cdot\mathbf{Y}})$, and 1-lattice means that $\mathbf{Y}$ takes values on the lattice L of points with integer co-ordinates, and no sub-lattice of L supports all $\mathbf{x} - \mathbf{y}$ with $P\{\mathbf{Y} = \mathbf{x}\} > 0$ and $P\{\mathbf{Y} = \mathbf{y}\} > 0$. In the 1-lattice case $f(\mathbf{t})$ has period 2π in each variable but $|f(\mathbf{t})| < 1$ when $\mathbf{t}$ is not of the form $(2m\pi, 2n\pi)$. The third term, by definition, means that $f(2m\pi, 0) = 1$ for $m = 0, \pm 1, \pm 2, \ldots$ but that $|f(\mathbf{t})| < 1$, otherwise.

Suppose that $\mathbf{Y}_n = (Y_n^{(1)}, Y_n^{(2)})$ are independent copies of $\mathbf{Y}$, that $\mathbf{S}_n = \sum_{m=1}^{n} \mathbf{Y}_m$, and that there exists $\mathbf{a}_n > \mathbf{0}$, $\mathbf{b}_n$ such that $\mathbf{X}_n \to^{D} \mathbf{X}$, where

$$\mathbf{X}_n = \{S_n^{(1)}/a_n^{(1)}, S_n^{(2)}/a_n^{(2)}\} - \mathbf{b}_n.$$

Then we say that $\mathbf{X}$ has a stable distribution and that $\mathbf{Y}$ belongs to the domain of attraction of $\mathbf{X}$. Since each component of $\mathbf{X}$ is clearly stable in the univariate sense, to each stable distribution in $\mathbb{R}^{(2)}$ there correspond two parameters, $0 < \alpha_1 \leqslant 2$, $0 < \alpha_2 \leqslant 2$, the indices of the marginal stable distributions. It is shown in [6] that if at least one of α_1, α_2 equals 2, then $\mathbf{X}$ has the distribution of a pair of independent stable distributions of indices α_1 and α_2, but that when both α_1 and α_2 are less than 2, matters are more complicated. Specifically, $\varphi(\mathbf{t}) = E(e^{i\mathbf{t}\cdot\mathbf{X}})$ is determined through its Lévy exponent by

$$\log \varphi(\mathbf{t}) = \int_{|\mathbf{u}| \geqslant 1} \{e^{i\mathbf{t}\cdot\boldsymbol{\tau}(\mathbf{u})} - 1\}\, \nu(d\mathbf{u})$$

$$+ \int_{0 < |\mathbf{u}| < 1} \{e^{i\mathbf{t}\cdot\boldsymbol{\tau}(\mathbf{u})} - 1 - i\mathbf{t}\cdot\boldsymbol{\tau}(\mathbf{u})\}\, \nu(d\mathbf{u}), \tag{2.1}$$

where, with $\delta_i = \alpha_i^{-1}$, $i = 1, 2$,

$$\boldsymbol{\tau}(\mathbf{u}) = \{|u^{(1)}|^{\delta_1} \operatorname{sign} u^{(1)}, |u^{(2)}|^{\delta_2} \operatorname{sign} u^{(2)}\}, \tag{2.2}$$

and ν is determined by

$$\nu\{\mathbf{u}: |\mathbf{u}| > r, \arg(\mathbf{u}) \in E\} = r^{-1} H(E), \tag{2.3}$$

H being a finite measure on $[0, 2\pi)$.

We need two preliminary results about $\psi(\mathbf{t}) := -\operatorname{Re}\{\log \varphi(t)\}$.

LEMMA 1. *If* $\mathbf{X}$ *has a stable* (α_1, α_2) *distribution and* $\mathbf{t}^*$ *denotes the vector with components* $2^{-\delta_i} t^{(i)}$, $i = 1, 2$, *then for all* $\mathbf{t}$,

$$\psi(\mathbf{t}) = 2\psi(\mathbf{t}^*). \tag{2.4}$$

Proof. When $\max(\alpha_1, \alpha_2) = 2$, (2.4) follows from result for univariate stable distributions. When $\max(\alpha_1, \alpha_2) < 2$, (2.1) gives

$$\psi(\mathbf{t}) = \int_{0 < |\mathbf{u}| < \infty} [1 - \cos\{t \cdot \boldsymbol{\tau}(\mathbf{u})\}]\, \nu(d\mathbf{u}), \tag{2.5}$$

and (2.4) follows by making the change of variables $v^{(1)} = 2u^{(1)}$, $v^{(2)} = 2u^{(2)}$. ∎

LEMMA 2. *If* $\mathbf{X}$ *has a non-degenerate stable* (α_1, α_2) *distribution then for some positive constant* c *and all* $\mathbf{t}$,

$$\psi(\mathbf{t}) \geqslant c\{|t^{(1)}|^{\alpha_1} + |t^{(2)}|^{\alpha_2}\}. \tag{2.6}$$

Proof. Again the case $\max(\alpha_1, \alpha_2) = 2$ follows from univariate results. When $\max(\alpha_1, \alpha_2) < 2$ we will assume, with no loss of generality, that $\alpha_1 \leqslant \alpha_2 < 2$, and show first that $\psi(\mathbf{t}) \neq 0$ for all $\mathbf{t} \neq \mathbf{0}$. For, by (2.5), $\psi(\mathbf{t}) = 0$ implies that ν is concentrated on $\{\mathbf{u}: \mathbf{t}\cdot\boldsymbol{\tau}(\mathbf{u}) = 2n\pi,\ n = 0, \pm 1, \pm 2, ...\}$; because of the radial symmetry of ν inherent in (2.3), this can only happen if ν is concentrated on $\{\mathbf{u}: \mathbf{t}\cdot\boldsymbol{\tau}(\mathbf{u}) = 0\}$. But in this case, we would also have

$\psi(a\mathbf{t})=0$ for any $a\neq 0$ which, in view of Lemma 3.1 of [7], contradicts the assumption of non-degeneracy. Since ψ is continuous, it follows that ψ is bounded away from 0 on any compact subset of $\mathbb{R}^{(2)}\setminus\{\mathbf{0}\}$. In particular, if $S(l)$ denotes the square of side-length l centered at $\mathbf{0}$, and $T=S(1)\setminus S(2^{-\delta_1})$, then $c=\frac{1}{2}\inf_{\mathbf{t}\in T}\psi(\mathbf{t})>0$, and (2.6) holds for $\mathbf{t}\in T$. If $\mathbf{t}\notin S(1)$ we define a sequence $\{\mathbf{t}_n, n\geqslant 1\}$ by setting $\mathbf{t}_1=\mathbf{t}$ and $\mathbf{t}_n=(\mathbf{t}_{n-1})^*$ for $n\geqslant 2$, and let $\hat{\mathbf{t}}=\mathbf{t}_{n^*}$, where $n^*=\min\{n:\mathbf{t}_n\in S(1)\}$. Then, since $\mathbf{t}_{n^*-1}\notin S(1)$ and $\delta_1\geqslant\delta_2$, $\hat{\mathbf{t}}\in T$ and by repeated use of (2.4), $\psi(\mathbf{t})=2^{n^*}\psi(\hat{\mathbf{t}})\geqslant 2^{n^*+1}c$. But from the definition of n^* we have $2^{n^*}\geqslant\max\{|t^{(1)}|^{\alpha_1}, |t^{(2)}|^{\alpha_2}\}\geqslant\frac{1}{2}\{|t^{(1)}|^{\alpha_1}+|t^{(2)}|^{\alpha_2}\}$ and (2.6) holds for $\mathbf{t}\notin S(1)$. If $\mathbf{t}\in S(2^{-\delta_1})$ and $\mathbf{t}\neq\mathbf{0}$, we define a sequence $\{\mathbf{t}_n, n\geqslant 1\}$ by setting $\mathbf{t}_1=\mathbf{t}$ and $(\mathbf{t}_n)^*=\mathbf{t}_{n-1}$ for $n\geqslant 2$, and let $\hat{\mathbf{t}}=\mathbf{t}_{n^*}$, where $n^*=\min\{n:\mathbf{t}_n\notin S(2^{-\delta_1})\}$. Then again $\hat{\mathbf{t}}\in T$, so that $\psi(\mathbf{t})=2^{-n^*}\psi(\hat{\mathbf{t}})\geqslant 2^{1-n^*}c\geqslant 2c\max\{|t^{(1)}|^{\alpha_1}, |t^{(2)}|^{\alpha_2}\}$, and (2.6) holds. ∎

An immediate implication of (2.6) is that $|\varphi(\mathbf{t})|$ is integrable over $\mathbb{R}^{(2)}$, and hence the well-known result that $\mathbf{X}$ has a continuous, bounded density follows. (This result, for the more general case of matrix normalization, is in [2] and seems originally to have been proved by Sharpe, the proof appearing in Hudson [5].)

We can now give our result in the lattice case.

THEOREM 1. *Suppose that* $\mathbf{Y}$ *has a 1-lattice distribution and* $\mathbf{X}_n\to^D\mathbf{X}$, *where* $\mathbf{X}$ *has a non-degenerate stable distribution with density* g. *Then uniformly for* $\mathbf{x}\in L$,

$$a_n^{(1)}a_n^{(2)}P\{\mathbf{S}_n=\mathbf{x}\}-g(\mathbf{x}_n)\to 0 \tag{2.7}$$

as $n\to\infty$, *where* $\mathbf{x}_n=\{x^{(1)}/a_n^{(1)}, x^{(2)}/a_n^{(2)}\}-\mathbf{b}_n$.

Proof. The argument proceeds along the same lines as the proof of the analogous result in the "narrow sense" case in [7]. Specifically, if we denote the LHS of (2.7) by E_n, standard inversion formulae give

$$4\pi^2E_n=\int_{-\pi a_n^{(1)}}^{\pi a_n^{(1)}}\int_{-\pi a_n^{(2)}}^{\pi a_n^{(2)}}\{\varphi_n(\mathbf{t})\}^n e^{-i\mathbf{t}\cdot\mathbf{x}_n}\,d\mathbf{t}-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(\mathbf{t})\,e^{-i\mathbf{t}\cdot\mathbf{x}_n}\,d\mathbf{t},$$

where $\varphi_n(\mathbf{t})=e^{-in^{-1}\mathbf{t}\cdot\mathbf{b}_n}f(t^{(1)}/a_n^{(1)}, t^{(2)}/a_n^{(2)})$, so that $\{\varphi_n(\mathbf{t})\}^n=E\{e^{i\mathbf{t}\cdot\mathbf{X}_n}\}$. We rewrite this as $4\pi^2E_n=\sum_1^4 I_r$, where

$$I_1=\int_{A_1}[\{\varphi_n(\mathbf{t})\}^n-\varphi(\mathbf{t})]\,e^{-i\mathbf{t}\cdot\mathbf{x}_n}\,d\mathbf{t},$$

$$I_r=\int_{A_r}\{\varphi_n(\mathbf{t})\}^n e^{-i\mathbf{t}\cdot\mathbf{x}_n}\,d\mathbf{t}\qquad\text{for}\quad r=2,3,$$

and

$$I_4 = -\int_{A_4} \varphi(\mathbf{t})\, e^{-i\mathbf{t}\cdot\mathbf{x}_n}\, d\mathbf{t}.$$

Here, with $R(a, b)$ denoting $\{\mathbf{t}: |t^{(1)}| \leqslant a, |t^{(2)}| \leqslant b\}$, $A_1 = R(\Delta, \Delta)$, $A_2 = R(da_n^{(1)}, da_n^{(2)})\backslash A_1$, $A_3 = R(\pi a_n^{(1)}, \pi a_n^{(2)})\backslash R(da_n^{(1)}, da_n^{(2)})$, and $A_4 = A_1^c$, where $0 < d < \pi$ and Δ are to be fixed later. Now $\{\varphi_n(\mathbf{t})\}^n \to \varphi(\mathbf{t})$ uniformly for $\mathbf{t} \in A_1$, so $I_1 \to 0$ uniformly in $\mathbf{x}$ as $n \to \infty$. Also $\rho := \sup_{\mathbf{t} \in A_3} |\varphi_n(\mathbf{t})| = \sup_{S(2\pi)\backslash S(2d)} |f(\mathbf{t})| < 1$, so $|I_3| \leqslant 4\pi^2 a_n^{(1)} a_n^{(2)} \rho^n$ and this $\to 0$ as $n \to \infty$. Obviously, by Lemma 2, we can make I_4 arbitrarily small by choosing Δ sufficiently large, so there remains only I_2. We estimate this by noting that

$$|I_2| \leqslant \sum_{m=0}^{k} \int_{C_m} \{|\varphi_n(\mathbf{t})|\}^n\, d\mathbf{t}, \tag{2.8}$$

where $C_m = A_2 \wedge \{R(\Delta 2^{(m+1)\delta_1}, \Delta 2^{(m+1)\delta_2})\backslash R(\Delta 2^{m\delta_1}, \Delta 2^{m\delta_2})\}$, and $k = \max\{k_1, k_2\}$, where $k_i = \min\{m: \Delta 2^{(m+1)\delta_i} > da_n^{(i)}\}$. Now as $n \to \infty$ $n \log |\varphi_n(\mathbf{t})| \to \operatorname{Re}\{\log \varphi(\mathbf{t})\} = -\psi(\mathbf{t})$, so by Lemma 1 and Lemma 2, $\log |\varphi_n(\mathbf{t})| / \log |\varphi_n(\mathbf{t}^*)| \to \psi(\mathbf{t})/\psi(\mathbf{t}^*) = 2$, uniformly on compact subsets of $\mathbb{R}^{(2)}\backslash\{\mathbf{0}\}$. Since $|\varphi_n(\mathbf{t})| = |f(t^{(1)}/a_n^{(1)}, t^{(2)}/a_n^{(2)})|$ and we know, by marginal considerations, that $a_n^{(i)} \to \infty$, $a_n^{(i)}/a_{n+1}^{(i)} \to 1$ for $i = 1, 2$, it follows that we can choose $d > 0$ such that $\log |f(\mathbf{s})| \leqslant \frac{3}{2} \log |f(\mathbf{s}^*)|$, and hence $|f(\mathbf{s})| \leqslant |f(\mathbf{s}^*)|^{3/2}$, whenever $\mathbf{s} \in S(d)$. Using this repeatedly, and then making the obvious change of variables gives

$$\begin{aligned}\int_{C_m} \{|\varphi_n(\mathbf{t})|\}^n\, d\mathbf{t} &\leqslant \int_{C_m} \left\{\left| f\left(\frac{2^{-m\delta_1} t^{(1)}}{a_n^{(1)}}, \frac{2^{-m\delta_2} t^{(2)}}{a_n^{(2)}}\right)\right|\right\}^{n(3/2)^m} d\mathbf{t} \\ &= 2^{m(\delta_1+\delta_2)} \int_{C_0} \{|\varphi_n(\mathbf{t})|\}^{n(3/2)^m}\, d\mathbf{t}.\end{aligned}$$

Now, from (2.6), we have $|\varphi(\mathbf{t})| \leqslant e^{-c\Delta^\alpha}$ for $\mathbf{t} \in C_0$, where $\alpha = \min(\alpha_1, \alpha_2)$, and, since $\{|\varphi_n(t)|\}^n \to |\varphi(\mathbf{t})|$ uniformly for $\mathbf{t} \in C_0$, it follows that for all large enough n, $\{|\varphi_n(\mathbf{t})|\}^n \leqslant 2e^{-c\Delta^\alpha}$. Thus, from (2.8),

$$\limsup_{n\to\infty} |I_2| \leqslant 4\Delta^2 2^{\delta_1+\delta_2} \sum_{m=0}^{\infty} 2^{m(\delta_1+\delta_2)} \{2e^{-c\Delta^\alpha}\}^{(3/2)^m},$$

which can clearly be made arbitrarily small by choosing Δ large enough. ∎

The corresponding result in the non-lattice case is

THEOREM 2. *Suppose that* $\mathbf{Y}$ *has a non-lattice distribution and* $\mathbf{X}_n \to^D \mathbf{X}$, *where* $\mathbf{X}$ *has a non-degenerate stable distribution with density g. Then*

$$a_n^{(1)} a_n^{(2)} P\{x^{(i)} \leqslant S_n^{(i)} \leqslant x^{(i)} + h^{(i)}, i = 1, 2\} - h^{(1)} h^{(2)} g(\mathbf{x}_n) \to 0 \tag{2.9}$$

as $n \to \infty$, *uniformly for* $\mathbf{x} \in \mathbb{R}^{(2)}$ *and* $\mathbf{h} \in$ *compact subsets of* $\mathbb{R}^+ \times \mathbb{R}^+$.

Proof. We omit details of this proof, as it follows the lines of the proof given in [8] for the "narrow sense" case, with fairly obvious modifications. Of course, as in [8], the crux is the estimate for I_2, which is the same as in Theorem 1. ∎

Finally we consider the jointly lattice and non-lattice case:

THEOREM 3. *Suppose that* $\mathbf{Y}$ *has a jointly* 1*-lattice and non-lattice distribution, and* $\mathbf{X}_n \to^D \mathbf{X}$, *where* $\mathbf{X}$ *has a non-degenerate stable distribution with density g. Then*

$$a_n^{(1)} a_n^{(2)} P\{S_n^{(1)} = x^{(1)},\, x^{(2)} \leqslant S_n^{(2)} \leqslant x^{(2)} + h\} - hg(\mathbf{x}_n) \to 0 \tag{2.10}$$

as $n \to \infty$, *uniformly for* $\mathbf{x} \in Z \times \mathbb{R}$ *and* $h \in$ *compact subsets of* $\mathbb{R}^+$.

Proof. Although this also follows the same lines as [8], certain non-trivial modifications are required. Explicitly, writing

$$P_n(\mathbf{x}, h) = a_n^{(1)} P\{S_n^{(1)} = x^{(1)},\, x^{(2)} \leqslant X_n^{(2)} \leqslant x^{(2)} + h\},$$

note that (2.10) will follow (by replacing h by $h/a_n^{(2)}$ and $x^{(2)}$ by $x_n^{(2)}$) if we can establish that, given any $\varepsilon > 0$, $N > 0$, $\exists n_0 > 0$, $h_0 > 0$, such that, with $\tilde{\mathbf{x}}_n = (x_n^{(1)}, x^{(2)})$,

$$|P_n(\mathbf{x}, h) - hg(\tilde{\mathbf{x}}_n)| \leqslant \varepsilon h, \tag{2.11}$$

for all $n \geqslant n_0$, $x^{(1)} \in Z$, $x^{(2)} \in \mathbb{R}$, and $(Na_n^{(2)})^{-1} \leqslant h \leqslant h_0$.

Now write $V_n(\mathbf{x}, h, a)$ for the convolution of $P_n(\mathbf{x}, h)$, as a function of $x^{(2)}$, with $K_a(x^{(2)})$, where $K_a(x) = 2a(\pi x^2)^{-1} (\sin \frac{1}{2}x)^2$, and let

$$I(\mathbf{x}, a, h) = h^{-1} V_n(\mathbf{x}, h, a) - \tilde{g}(\mathbf{x}_n).$$

Then, in almost exactly the way that Lemma 2 of [8] follows from Lemma 1 of [8], it can be seen that (2.11) will follow if we can show that as $n \to \infty$, $h \to 0$, and $a \to 0$, subject to $a \geqslant (Na_n^{(2)})^{-1}$, $I(\mathbf{x}, h, a) \to 0$ uniformly in $\mathbf{x}$. Since the Fourier transform of K_a is $\{1 - a\,|t|\}^+$ and the Fourier transform of P_n is $a_n^{(1)}(it)^{-1}(1 - e^{-iht})\, E\{e^{itX_n^{(2)}}; S_n^{(1)} = x^{(1)}\}$, we have

$$V_n(\mathbf{x}, h, a) = \frac{ha_n^{(1)}}{2\pi} \int_{-a^{-1}}^{a^{-1}} e^{-itx^{(2)}} H(t)\, E\{e^{itX_n^{(2)}}; S_n^{(1)} = x^{(1)}\}\, dt, \tag{2.12}$$

where $H(t) = \{1 - a\,|t|\}^+ (iht)^{-1} \{1 - e^{-iht}\}$. Forming the Fourier series

$\sum_{x^{(1)}=-\infty}^{\infty} V_n(\mathbf{x}, h, a)\, e^{ix^{(1)}t^{(1)}}$, applying an inversion formula, and making a change of variable, we can rewrite (2.12) as

$$V_n(\mathbf{x}, h, a) = \frac{h}{4\pi^2} \int_{-\pi a_n^{(1)}}^{\pi a_n^{(1)}} \int_{-a^{-1}}^{a^{-1}} e^{-i\mathbf{t}\cdot\tilde{\mathbf{x}}_n} H(t^{(2)}) \{\varphi_n(\mathbf{t})\}^n \, d\mathbf{t}. \tag{2.13}$$

Thus $4\pi^2 I = \sum_1^4 I_r$, where

$$I_1 = \int_{A_1} e^{-i\mathbf{t}\cdot\tilde{\mathbf{x}}_n} [H(t^{(2)})\{\varphi_n(\mathbf{t})\}^n - \varphi(\mathbf{t})]\, d\mathbf{t}, \qquad A_1 = S(\Delta),$$

$$I_r = \int_{A_r} e^{-i\mathbf{t}\cdot\tilde{\mathbf{x}}_n} H(t^{(2)})\{\varphi_n(\mathbf{t})\}^n \qquad \text{for} \quad r = 2, 3$$

where

$$A_2 = R(da_n^{(1)}, da_n^{(2)}) \backslash A_1, \qquad A_3 = R(\pi a_n^{(1)}, a^{-1}) \backslash R(da_n^{(1)}, da_n^{(2)}),$$

and

$$I_4 = -\int_{A_4} e^{-i\mathbf{t}\cdot\tilde{\mathbf{x}}_n} \varphi(\mathbf{t})\, d\mathbf{t}, \qquad A_4 = A_1^c,$$

where $0 < d < N$ and Δ are to be fixed later. Now for $|t^{(2)}| \leqslant a^{-1}$, $|H(t^{(2)})| \leqslant 1$, so $|I_2| \leqslant \int_{A_2} \{|\varphi_n(\mathbf{t})|\}^n \, d\mathbf{t}$ and, by the argument used in the proof of Theorem 1, both $|I_4|$ and $\limsup_{n\to\infty} |I_2|$ can be made arbitrarily small by choosing d sufficiently small and Δ sufficiently large. Also, using $a^{-1} \leqslant N a_n^{(2)}$, we have

$$|I_3| \leqslant \int_{A_3} \{|\varphi(\mathbf{t})|\}^n \, d\mathbf{t} \leqslant a_n^{(1)} a_n^{(2)} \int_B \{|f(\mathbf{s})|\}^n \, d\mathbf{s} \leqslant 4\pi N \rho^n a_n^{(1)} a_n^{(2)},$$

where $B = R(\pi, N) \backslash S(d)$ so that $\rho := \sup_{\mathbf{s}\in B} |f(\mathbf{s})| < 1$ by the jointly 1-lattice and non-lattice assumption. Thus $I_3 \to 0$ as $n \to \infty$, subject to $a^{-1} \leqslant N a_n^{(2)}$ and, since $\{\varphi_n(\mathbf{t})\}^n \to \varphi(\mathbf{t})$ as $n \to \infty$ and $H(t^{(2)}) \to 1$ as $h \to 0$ and $a \to 0$, uniformly on A_1, $I_1 \to 0$. Thus $I \to 0$ as required, and (2.11) and hence (2.10) follow. ∎

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